Lesson 34: Exponential Growth

Ex 1 Find the function passing through $(0,5)$ with

$\frac{dy}{dt} = 3y$ ← diff eq.   initial value

OPTION 1:

$\frac{d}{dt}[e^{3t}] = 3e^{3t}$

$y = Ce^{3t}$ $(y' = 3 \cdot Ce^{3t} = 3y)$

OPTION 2: (Separation of Variables) ← don't need to know!

$\frac{dy}{dt} = 3y$

$\int \frac{1}{y}\,dy = \int 3\,dt$

$\ln|y| = 3t + D$

$\ln y = 3t + D$ (if $y \geq 0$)

$y = e^{3t+D}$

$y = e^{3t}\cancel{e^{D}} \to C$

$y = Ce^{3t}$

at $(0,5)$: $5 = C\cancel{e^{3(0)}} \to 1$

$C = 5$

$\boxed{y = 5e^{3t}}$

Fact If $\frac{dy}{dt} = ky$ (where k is a constant), then $y = Ce^{kt}$.

growth rate/ proportionality constant

initial value

If $k > 0$, then y is increasing, and this is called <u>exponential growth</u>.

Ex 2 The rate of change of P is proportional to P. $\rightarrow \frac{dP}{dt} = kP$
When $t=1$, $P=400$, and when $t=2$, $P=600$.

(a) What does $\frac{dP}{dt}$ equal?

$$\frac{dP}{dt} = kP$$

(b) Find $P(t)$.

$$P = Ce^{kt} \quad \text{(by Fact)}$$

$$\begin{cases} t=1: & 400 = Ce^{k} \quad ① \\ t=2: & 600 = Ce^{2k} \quad ② \end{cases}$$

$$\frac{②}{①}: \frac{600}{400} = \frac{\cancel{C}e^{2k}}{\cancel{C}e^{k}} \qquad \frac{e^{2k}}{e^{k}} = e^{2k-k}$$

$$\frac{3}{2} = e^{2k-k} \quad \text{(exponent rule)}$$

$$\frac{3}{2} = e^{k}$$

$$k = \ln\left(\frac{3}{2}\right)$$

Back substitute to solve for C:

$$① \quad 400 = Ce^{\ln(3/2)}$$

$$C = \frac{400}{e^{\ln(3/2)}} = \frac{400}{3/2} = 400 \cdot \frac{2}{3} = \frac{800}{3}$$

$$\text{So } P = \frac{800}{3}\left(e^{\ln(3/2)t}\right) \quad \text{(exponent rule)}$$

$$= \frac{800}{3}\left(\left(\cancel{e^{\ln}}(3/2)\right)^{t}\right)$$

$$= \boxed{\frac{800}{3}\left(\frac{3}{2}\right)^{t}}$$

Ex 3 If $P'(t)$ is proportional to $P(t)$ with a growth rate of 3, $k=3$
and $P(10) = 100$, find $P(t)$. $\hookrightarrow \frac{dP}{dt} = 3P$

$$P(t) = Ce^{3t} \quad \text{(Fact)}$$

$$\text{at } t=10: \quad 100 = Ce^{3(10)}$$

$$C = \frac{100}{e^{30}}$$

$$P(t) = \frac{100}{e^{30}} e^{3t} = \boxed{100e^{3t-30}}$$

The formula for computing continually compounded interest is $\boxed{A = Pe^{rt}}$ where:

A = amount after t years
P = initial (principal) amount
r = annual interest rate (as a decimal) $(1\% \Rightarrow r = .01)$
t = number of years

	Initial Investment (P)	Annual Rate (r)	Time to Double	Amount after 5 years (A) (t=5)
(a)	\$400	3%	$\boxed{23.1\text{ yrs}}$	$\boxed{464.73}$
(b)	\$400	$\boxed{5.78\%}$	12	$\boxed{533.94}$
(c)	\$400	$\boxed{11.19\%}$	$\boxed{6.19\text{yrs}}$	\$700
(d)	$\boxed{\$602.50}$	3%	$\boxed{23.1\text{yrs}}$	\$700

(a) $A = Pe^{rt}$
$A = 400e^{.03t}$
$A = 400e^{.03(5)} \approx \$464.73$

$800 = 400e^{.03t}$
$2 = e^{.03t}$
$\ln 2 = .03t$
$t = \frac{\ln 2}{.03} \approx 23.10$

Note $\boxed{\text{Time to double} = \frac{\ln 2}{r}}$ or $r = \frac{\ln 2}{\text{time to double}}$

(b) $r = \frac{\ln 2}{12} \approx .05776$

(c) $A = Pe^{rt}$
$700 = 400e^{r(5)}$
$\frac{7}{4} = e^{5r}$
$\ln\frac{7}{4} = 5r$
$r = \frac{\ln\frac{7}{4}}{5} \approx .11192$

(d) $A = Pe^{rt}$
$700 = Pe^{.03(5)}$
$P = \frac{700}{e^{.03(5)}} \approx 602.50$

Time to double $= \frac{\ln 2}{.03} \approx 23.1$